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Conservation laws for conformally invariant variational problems

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Abstract. We succeed in writing 2-dimensional conformally invariant non-linear elliptic PDE (harmonic map equation, prescribed mean curvature equations, . . . , etc.) in divergence form. These divergence-free quantities generalize to target manifolds without symmetries the well known conservation laws for weakly harmonic maps into homogeneous spaces. From this form we can recover, without the use of moving frame, all the classical regularity results known for 2-dimensional conformally invariant non-linear elliptic PDE (see [Hel]). It enables us also to establish new results. In particular we solve a conjecture by E. Heinz asserting that the solutions to the prescribed bounded mean curvature equation in arbitrary manifolds are continuous and we solve a conjecture by S. Hildebrandt [Hil1] claiming that critical points of continuously differentiable elliptic conformally invariant Lagrangian in two dimensions are continuous.

I Introduction

The absence of possible applications of the maximum principle to solutions to non-linear elliptic systems reduces drastically the tools available for answering questions regarding the symmetry, the uniqueness or the regularity of these solutions. In such an impoverishment of the available technics while passing from scalar PDE to systems, the search for conservation laws is, however, one of the remaining relevant strategy to adress these questions. Weakly harmonic maps into spheres give a good illustration of the efficiency of conservation laws in this setting. An weakly harmonic map u from the n-dimensional unit ball B^n into the unit sphere S^{m-1} of $\mathbb{R}^m$ is a $W^{1,2}(B^n, \mathbb{R}^m)$ map (maps in L^2 whose first derivatives are also in L^2) which takes values almost everywhere in the sphere S^{m-1} and which solves the following PDE

$$-\Delta u = u\,|\nabla u|^2 \tag{I.1}$$

where Δ is the negative laplacian in $\mathbb{R}^n$: $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. They are the critical points of the Dirichlet energy $E(u) = \int_{B^n} |\nabla u|^2\, dx_1 \cdots dx_n$ for all perturbations of the form $u_t = u - t\phi/|u - t\phi|$ where ϕ is an arbitrary compactly supported smooth map from B^n into $\mathbb{R}^m$. Because of the conformal invariance of the Dirichlet energy E in 2 dimension ($E(u\circ\varphi) = E(u)$ for arbitrary u in $W^{1,2}(\mathbb{R}^2, \mathbb{R}^m)$ and arbitrary conformal map φ from $\mathbb{R}^2$ into $\mathbb{R}^2$), the harmonic map equation (I.1) is conformally invariant in 2 dimension: if u is a solution to (I.1) in $W^{1,2}(B^2, \mathbb{R}^m)$, the composition with an arbitrary conformal map φ: $u \circ \varphi$ is again a solution to (I.1). The conformal dimension 2 is also the critical dimension for (I.1): The left-hand-side of (I.1) for a $W^{1,2}$ solution is in L^1, therefore a solution has a laplacian in L^1 which is the borderline case in 2 dimension which "almost" ensures that the first derivatives are in L^2 (using standard estimates on Riesz potential [Ste]). So, in some sense, by inserting the $W^{1,2}$ bound assumption in the non-linearity we are almost back on our feet by bootstrapping this regularity information in the linear part of the equation. None of the two sides, linear and non-linear, of the equation is really dominant: the equation is critical. Solution to this system enters in the family of solutions of systems of quadratic growth and can be discontinuous (see for instance [Gia]). Hence, among the fundamental analysis issues regarding (I.1) are **1)** the regularity of solution in conformal 2-dimension and **2)** the passage to the limit in the equation for sequences of solutions having bounded E energy. Both questions were solved by the introduction of the following conservation laws discovered by J. Shatah ([Sha]): u is a solution to (I.1) in $W^{1,2}$ if and only if the following holds

$$\forall i, j \in \{1, \dots, m\} \quad div(u^i \nabla u^j - u^j \nabla u^i) = 0. \tag{I.2}$$

The cancellation of these divergences can be interpreted by the mean of Noether theorem using the symmetries of the target S^{m-1} (see Hélein's book [Hel]). Using this form it becomes straightforward to answer to the question **2)** using the compactness of the embedding of $W^{1,2}$ into L^2 (Rellich Kondrachov embedding Theorem). The answer to question **1)**: the fact that $W^{1,2}$ solutions to (I.1) are real analytic was established by F. Hélein (see [Hel]) starting again from the conservation laws (I.2). The main step was to prove the continuity of the solution since, by classical results in [HiW,LaU] and [Mo], continuous solutions are real analytic. Using the conservation laws (I.2) and the fact that $\sum_j u^j \nabla u^j = 0$, F. Hélein wrote (I.1) in the following way:

$$\begin{aligned} -\Delta u^i &= \sum_{j=1} u^i\, \nabla u_j \cdot \nabla u^j = \sum_{j=1}^{m} \left[u^i\, \nabla u_j - u_j\, \nabla u^i\right] \cdot \nabla u^j \\ &= \sum_{j=1}^{n} \nabla^\perp B^i_j \cdot \nabla u^j, \end{aligned} \tag{I.3}$$

where $\nabla^{\perp}$ is the ∇-operator rotated by $\pi/2$ (i.e. $\nabla^{\perp} := (-\partial_{x_2}, \partial_{x_1})$. The existence of B^i_j in $W^{1,2}_{loc}$ solving $curl B^i_j = u^i \nabla u_j - u_j \nabla u^i$ is given by the classical theory of elliptic operators. The product curl-grad in the right-hand-side of (I.3) has in fact some additional regularity than being simply in L^1 and the inverse by the laplace operator of such a product is continuous. This special phenomenon that we recall in the appendix was first observed in a particular case in [We] by H. Wente and was proved in its full generality by H. Brezis and J.M. Coron in [BrC] extending Wente's argument and independently, using a quite different approach, by L. Tartar in [Ta1]. Later on, the product curl-grad was observed to be in the local Hardy space $\mathcal{H}^1_{loc}$, smaller than L^1, by R. Coifman, P.L. Lions, Y. Meyer and S. Semmes in [CLMS] following the work of S. Müller [Mu] where this result was obtained under some sign assumption on the product. Among the special features of distributions in $\mathcal{H}^1_{loc}$ is the "nice" behavior of this space with respect to Calderon–Zygmund operators, in particular the inverse of such a distribution by the Laplace operator are in $W^{2,1}$ which embeds in C^0 in 2 dimension. Observing that the non-linearity of the weakly harmonic map equation is in $\mathcal{H}^1_{loc}$ gives not only another approach to conclude that solutions to (I.1) are smooth in 2 dimension but also permits to establish the estimate

$$\int_O |\nabla^2 u| \, dx_1 \cdots dx_n < +\infty, \tag{I.4}$$

for any solution to (I.1), for n arbitrary and where O is an arbitrary open subset with closure in B^n. This estimate happens to play a crucial role for establishing energy quantization results as described in [LiR].

It is now natural to try to understand to which extend the above results are still valid when we are considering $W^{1,2}$ weakly harmonic maps taking values in an arbitrary submanifold of $\mathbb{R}^m$. What about questions **1)**, **2)** or what about the validity of the estimate (I.4) in this general setting? Let then, N^k be a C^2 k-dimensional submanifold of $\mathbb{R}^m$. Denote π_N the C^1 orthogonal projection on N^k defined in a small tubular neighborhood of N^k in $\mathbb{R}^m$ which assigns to each point in this neighborhood the nearest point on N. We denote by $W^{1,2}(B^n, N^k)$ the subset of $W^{1,2}$ maps from B^n into $\mathbb{R}^m$ which take values in N^k almost everywhere. The critical points u in $W^{1,2}(B^n, N^k)$ of the Dirichlet energy $E(u) = \int_{B^n} |\nabla u|^2$ for all perturbations of the form $\pi_N(u + t\phi)$, where ϕ is an arbitrary smooth compactly supported map from B^n into $\mathbb{R}^m$, are the weakly harmonic maps from B^n into N^k. They are the maps in $W^{1,2}(B^n, N^k)$ which solve the following Euler–Lagrange equation (in distributional sense)

$$-\Delta u = A(u)(\nabla u, \nabla u) = \sum_{l=1}^{n} A(u)(\partial_{x_l} u, \partial_{x_l} u) = 0, \tag{I.5}$$

where $A(u)$ is the second fundamental form at $u(x)$ for the submanifold N^k in $\mathbb{R}^m$. For instance for $k = m - 1$ when N^{m-1} is an oriented codimension 1 submanifold, if we denote by $n(y)$ the Gauss map of N^{m-1} at y, the unit perpendicular vectorfields which generates the orientation of N^{m-1}, (I.5) becomes

$$-\Delta u = n\,\nabla n \cdot \nabla u, \tag{I.6}$$

where we keep denoting n the composition $n \circ u$. In order to try to extend the above described results established for $W^{1,2}$ solutions to (I.1) to solutions of (I.6), or even more generally to solutions to (I.5), it is then natural to look for conservation laws (divergence free quantities) generalizing (I.2). Is for instance the non-linearity $n\,\nabla n \cdot \nabla u$ in the right-hand-side of (I.6) (or even (I.5) in the local Hardy space $\mathcal{H}^1_{loc}$? Do we have estimates of the form (I.4) for general $W^{1,2}$ solutions to (I.6) or even (I.5)? **Can we write the equation (I.6) or (I.5) in divergence form?** Until now the answers to these questions were open and only the introduction of the indirect but beautiful technic of moving frame by F. Hélein permitted to avoid the *direct conservation law approach* for proving questions like the regularity in dimension 2 of the weakly harmonic maps into general target (i.e. solutions to (I.6)) – see again [Hel]. This set of questions has motivated the following, which is one of the main result of the present paper:

Theorem I.1 *Let $m \in \mathbb{N}$. For every $\Omega = (\Omega^i_j)_{1\le i,j\le m}$ in $L^2(D^2, so(m)\otimes\mathbb{R}^2)$ (i.e. $\forall i, j \in \{1, \dots, m\}$, $\Omega^i_j \in L^2(D^2, \mathbb{R}^2)$ and $\Omega^i_j = -\Omega^j_i$), every $u \in W^{1,2}(D^2, \mathbb{R}^m)$ solving*

$$-\Delta u = \Omega \cdot \nabla u \tag{I.7}$$

is continuous where the contracted notation in (I.7) using coordinates stands for $\forall i = 1, \dots, m$ $-\Delta u^i = \sum_{j=1}^m \Omega^i_j \cdot \nabla u^j$.

This theorem is optimal because of the following counterexample of Frehse [Fre]: for $m = 2$ the map $u = (u_1, u_2)$ from D^2 into $S^1 \subset \mathbb{R}^2$ defined by

$$u_1(x) = \sin\log\log\frac{2}{|x|}, \quad u_2(x) = \cos\log\log\frac{2}{|x|}, \tag{I.8}$$

is in $W^{1,2}(D^2, \mathbb{R}^2)$ solves (I.7) for

$$\Omega = \begin{pmatrix} (u_1+u_2)\nabla u_1 & (u_1+u_2)\nabla u_2 \\ (u_2-u_1)\nabla u_1 & (u_2-u_1)\nabla u_2 \end{pmatrix} \in L^2\big(D^2, M_2 \otimes \mathbb{R}^2\big), \tag{I.9}$$

but Ω is not antisymmetric and the solution is L^∞ but not continuous. One can even find counterexamples for Ω non antisymmetric and for which u is not in L^∞: for $m = 2$ the map $u = (u_1, u_2)$ from D^2 into $\mathbb{R}^2$ given by

$$u_1(x) = \log\log\frac{2}{|x|}, \qquad u_2(x) = \log\,\log\frac{2}{|x|}, \tag{I.10}$$

is in $W^{1,2}(D^2, \mathbb{R}^2)$ and solves (I.7) for

$$\Omega = \begin{pmatrix} \nabla u_1 & 0 \\ 0 & \nabla u_2 \end{pmatrix} \in L^2\big(D^2, M_2 \otimes \mathbb{R}^2\big), \tag{I.11}$$

but Ω is symmetric and not antisymmetric.

Theorem I.1 applies to (I.6) because of the following observation: every derivative of u solving (I.6) is tangent to N^{m-1} and is therefore perpendicular to n. Thus $\sum_{j=1}^m n_j \nabla u^j = 0$ and we can rewrite (I.6) in the form:

$$-\Delta u^i = \sum_{j=1}^m \big[n^i \nabla n_j - n_j \nabla n^i\big] \cdot \nabla u^j. \tag{I.12}$$

Taking now $\Omega^i_j := n^i \nabla n_j - n_j \nabla n^i$ we can apply Theorem I.1 to get the continuity of u. This way of rewriting the equation has to be compared with the particular case (I.3) except that in the general case there is no reason for $\Omega^i_j := n^i \nabla n_j - n_j \nabla n^i$ to be divergence free. One of the main observation of the present work is that what is important in (I.3) is not the divergence free structure of $n^i \nabla n_j - n_j \nabla n^i$, valid in the particular case of the round sphere and which disappears as soon as one perturbs the metric of the target, but it is the anti-symmetry of this quantity which is much more robust and which is the key point for the regularity of solution to (I.6). This is a new compensation phenomenon that we discovered, which goes beyond the curl-grad structures although it is strongly linked to it as we will explain in the paper. In fact we observed that not only solutions to (I.6), not only solutions to (I.5) but every critical point of any elliptic conformally invariant Lagrangian in dimension 2 can be written in the form (I.7) and the regularity result obtained in Theorem I.1 can be applied to them. Precisely we have

Theorem I.2 *Let N^k be a C^2 submanifold of $\mathbb{R}^m$ (k and m being arbitrary integer satisfying $1 \leq k \leq m$). Let ω be a C^1 $2-$form on N^k such that the L^∞ norm of $d\omega$ is bounded on N^k. Then every critical point in $W^{1,2}(D^2, N^k)$ of the Lagrangian*

$$F(u) = \int_{D^2} \big[|\nabla u|^2 + \omega(u)(\partial_x u, \partial_y u)\big]\, dx \wedge dy \tag{I.13}$$

satisfies an equation of the form (I.7) for some Ω in $L^2(D^2, so(m) \otimes \mathbb{R}^2)$ and is therefore continuous.

Critical points of F which are conformal are immersed discs in N^k whose mean curvature in N^k at u is given by $|\nabla u|^{-2} d\omega(u)(\cdot, \partial_x u, \partial_y u)$. This is the so called prescribed mean curvature equation in a manifold N^k. It is not difficult to see that the Lagrangian of the form (I.13) are conformally invariant. Conversely, It was proved in [Gr1] that every conformally invariant elliptic Lagrangian, satisfying some “natural conditions”, generates an

Euler–Lagrange equation corresponding to a prescribed mean curvature equation in a manifold. A particular case of interest is the case $k = m = 3$ and $N^3 = \mathbb{R}^3$. Denote $2d\omega = H(z)dz_1dz_2dz_3$ the Euler–Lagrange equation to F in that case is

$$-\Delta u = -2H(u)\,\partial_x u \wedge \partial_y u. \tag{I.14}$$

There has been several attempts to prove the continuity of solutions to (I.14) under several assumptions on H like $\|H\|_\infty + \|\nabla H\|_\infty < +\infty$ (see for instance [Hei1,Hei2,Gr2,Bet1,BeG1,BeG2,Cho]). It was conjectured by E. Heinz, see [Hei3], that the weakest possible assumption $\|H\|_{L^\infty(\mathbb{R}^3)} < +\infty$ should suffices to ensure the continuity of $W^{1,2}$ solutions to (I.14) and that no control of any kind of the differentiability of the prescribed mean curvature H was needed. By denoting $\nabla^\perp := (-\partial_y, \partial_x)$ and introducing

$$\Omega := H(u)\begin{pmatrix} 0 & \nabla^\perp u^3 & -\nabla^\perp u^2 \\ -\nabla^\perp u^3 & 0 & \nabla^\perp u^1 \\ \nabla^\perp u^2 & -\nabla^\perp u^1 & 0 \end{pmatrix} \tag{I.15}$$

Equation (I.14) becomes of the form

$$-\Delta u = \Omega \cdot \nabla u,$$

where $\Omega \in L^2(D^2, so(3)\otimes\mathbb{R}^2)$. We can then apply Theorem I.1 to (I.14) and we have then proved Heinz's conjecture on prescribed mean curvature equations. In fact Theorem I.2 solves a conjecture by S. Hildebrandt claiming that the critical points of the second order C^1 elliptic conformally invariant lagrangian in 2 dimensions are continuous see [Hil1] and [Hil2] (3.15). In [Hel1] the class of general conformally invariant lagrangian in 2 dimension is analyzed.

Theorem I.1 is based on the discovery of conservation laws generalizing (I.2). Denoting $M_m(\mathbb{R})$ the space of square $m \times m$ real matrices, we have:

Theorem I.3 *Let $m \in \mathbb{N}$. Let $\Omega = (\Omega^i_j)_{1\le i,j\le m}$ in $L^2(B^n, so(m) \otimes \wedge^1\mathbb{R}^n)$ and let $A \in L^\infty(B^n, M_m(\mathbb{R})) \cap W^{1,2}$ and $B \in W^{1,2}(B^n, M_m(\mathbb{R}) \otimes \wedge^2\mathbb{R}^n)$ satisfying*

$$d_\Omega A := dA - A\Omega = -d^*B \tag{I.16}$$

(where explicitly (I.18) *means $\forall i, j \in \{1, \dots, m\}$ $dA^i_j - \sum_{k=1}^m A^i_k\Omega^k_j = -d^*B^i_j$). Then every solution to* (I.7) *on B^n satisfies the following conservation law*

$$d\left(*A\,du + (-1)^{n-1}(*B) \wedge du\right) = 0. \tag{I.17}$$

For $n = 2$, using different notations, the theorem says that given $\Omega = (\Omega^i_j)_{1\le i,j\le m}$ in $L^2(D^2, so(m)\otimes\mathbb{R}^2)$, $A \in L^\infty(D^2, M_m(\mathbb{R}))\cap W^{1,2}$ and $B \in W^{1,2}(D^2, M_m(\mathbb{R}))$ satisfying

$$\nabla_\Omega A := \nabla A - A\Omega = \nabla^\perp B. \tag{I.18}$$

Then every solution to (I.7) satisfies the following conservation law

$$div(A\nabla u + B\nabla^\perp u) = 0. \tag{I.19}$$

For instance going back to the symmetric situation of weakly harmonic maps into S^{m-1} we take A and B satisfying

$$\begin{cases} A = id_m = \left(\delta^j_i\right)_{1\le i,j\le m}, \\ \nabla^\perp B^i_j = u^i\nabla u_j - u_j\nabla u^i. \end{cases}$$

Using now the fact that $div(B^i_j\nabla^\perp u^j) = \nabla B^i_j\cdot\nabla^\perp u^j = -\nabla^\perp B^i_j\cdot\nabla u^j$ the harmonic map equation into S^{m-1}, with these notations, is equivalent to (I.19). We then have included the classical conservation law (I.2) into the larger set of conservation laws of the form (I.19). The question remains of finding A and B satisfying (I.18). We shall prove the following local existence result

Theorem I.4 *There exists $\varepsilon(m) > 0$ and $C(m)$ such that, for every $\Omega = (\Omega^i_j)_{1\le i,j\le m}$ in $L^2(D^2, so(m)\otimes\mathbb{R}^2)$ satisfying*

$$\int_{D^2} |\Omega|^2 \le \varepsilon_m, \tag{I.20}$$

there exists $A \in L^\infty(D^2, Gl_m(\mathbb{R}))\cap W^{1,2}$ and $B \in W^{1,2}(D^2, M_m(\mathbb{R}))$ satisfying

i)

$$\int_{D^2} |\nabla A|^2 + |\nabla A^{-1}|^2 + \|dist(A, SO(n))\|^2_\infty \le C(n)\int_{D^2}|\Omega|^2, \tag{I.21}$$

ii)

$$\int_{D^2}|\nabla B|^2 \le C(n)\int_{D^2}|\Omega|^2, \tag{I.22}$$

iii)

$$\nabla_\Omega A := \nabla A - A\Omega = \nabla^\perp B. \tag{I.23}$$

A corresponding local existence result in higher dimension is still an open problem. If the weakly harmonic map we are considering is stationary (see [Hel]), Ω is in the Morrey space given by

$$\|\Omega\|_{M_2^2} = sup_{x,r} \frac{1}{r^{n-2}} \int_{B_r(x)} |\Omega|^2 < +\infty. \tag{I.24}$$

Then, under the assumption that $\|\Omega\|_{M_2^2}$ is below some positive constant depending only on n and m, the elliptic linear system (I.16) becomes critical and the existence of A and B solving (I.16) should be looked for in the space M_2^2 (following the search of a Coulomb gauge in Morrey spaces introduced in [MeR], one has a replacement of Lemma A.3 in higher dimension).

Local existence of conservation law (I.19) for stationary weakly harmonic maps permits to extend to general C^2 targets the partial regularity of L.C. Evans [Ev] for weakly harmonic maps into spheres following the same strategy that Evans introduced.

This program is partly realised in a work in preparation of the author together with Michael Struwe.

Using conservation laws (I.19) we can prove the following result.

Theorem I.5 *Let $\Omega_n \in L^2(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ such that Ω_n weakly converges in L^2 to some Ω. Let f_n be a sequence in $H^{-1}(D^2, \mathbb{R}^m)$ which converges to 0 in H^{-1} and u_n be a bounded sequence in $W^{1,2}(D^2, \mathbb{R}^m)$ solving*

$$-\Delta u_n = \Omega_n \cdot \nabla u_n + f_n \qquad \text{in } D^2. \tag{I.25}$$

Then, there exists a subsequence $u_{n'}$ of u_n which weakly converges in $W^{1,2}$ to a solution of (I.7).

Passage to the limit in the equation in 2 dimension for the prescribed mean curvature equation or for the harmonic map equation was established in [Bet2] using involved technics. A much simpler proof using moving frames was then given in [FMS]. In both proofs a Lipschitz bound on the prescribed mean curvature was required. This is no more the case in Theorem I.5 where only an L^∞ bound on the prescribed mean curvature is needed.

Following similar ideas Theorem I.1 can be extended in its spirit to first order elliptic complex valued PDE's

Theorem I.6 *Let $m \in \mathbb{N}$ and $k \in \mathbb{N}$. Let $\Omega = (\Omega^i_j)_{1\le i,j\le m}$ in $L^2(D^2, so(m) \otimes \mathbb{C} \otimes \wedge^1\mathbb{R}^2)$ and let $\alpha \in L^2(D^2, M_{m,k}(\mathbb{C}))$ solving*

$$\frac{\partial \alpha}{\partial \overline{z}} = \Omega\, \alpha, \tag{I.26}$$

then there exists $P \in W^{1,2}(D^2, SO_m(\mathbb{C}))$ and $\beta \in C^\infty(D^2, M_{m,k}(\mathbb{C}))$ such that $\alpha = P\beta$, where $SO_m(\mathbb{C})$ is the group of invertible matrices in $Gl_m(\mathbb{C})$ satisfying $P^t P = id_m$.

Conservation laws and moving frames. Existence of global conservation laws can be obtained in the same spirit by the mean of moving frames. Considering a map u in $W^{1,2}(D^2, N^2)$ where N^2 is a closed C^2 oriented 2-dimensional submanifold of $\mathbb{R}^m$, then there exists a map e_1 in $W^{1,2}(D^2, S^{m-1})$ such that $e_1(x) \in T_{u(x)}N^2$ for almost every x in D^2. Moreover, denoting $e_2(x)$ the unit vector perpendicular to e_1 such that $e_1 \wedge e_2$ is the unit 2-vector giving the oriented tangent plane $T_{u(x)}N^2$, we can choose e_1 such that $div((e_2, \nabla e_1)) = 0$ on D^2 (see [Hel]) where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^m$ that we also sometime simply denote $\cdot$. Such a pair (e_1, e_2) is called a Coulomb moving frame associated to u. We have then the following conservation law

Theorem I.7 *Let u be a $W^{1,2}$ weakly harmonic map from D^2 into N^2, let (e_1, e_2) be a Coulomb moving frame associated to u, let a be the function solving*

$$\begin{cases} -\Delta a = \left(\nabla^{\perp} e_1, \nabla e_2\right) = \dfrac{\partial e_1}{\partial x} \cdot \dfrac{\partial e_2}{\partial y} - \dfrac{\partial e_1}{\partial y} \cdot \dfrac{\partial e_2}{\partial x} & \text{in } D^2, \\ a = 0 & \text{on } \partial D^2. \end{cases} \tag{I.27}$$

then the following conservation law holds

$$\begin{cases} div\left(\cosh a\,(\nabla u, e_1) + \sinh a\,(\nabla^{\perp} u, e_2)\right) = 0, \\ div\left(\cosh a\,(\nabla u, e_2) - \sinh a\,(\nabla^{\perp} u, e_1)\right) = 0. \end{cases} \tag{I.28}$$

Moreover the following estimate holds

$$\int_{D^2_{1/2}} |\nabla^2 u| \leq C \exp\left[\frac{1}{4\pi}\int_{D^2} |\nabla e|^2\right] \left(\|\nabla e\|_{L^2(D^2)} + 1\right) \|\nabla u\|_{L^2(D^2)}, \tag{I.29}$$

where $|\nabla e|^2 := |\nabla e_1|^2 + |\nabla e_2|^2$.

Observe that, because of Wente's Lemma A.1 that we recall in the appendix, the solution a of (I.27) is bounded in L^∞. When N^2 is not diffeomorphic to S^2 one can estimate $\int_{D^2} |\nabla e|^2$ in terms of $\|u\|_{W^{1,2}}$. This is no more the case for $N^2 = S^2$: one can find sequences of u_n, weakly harmonic from D^2 into S^2 with uniformly bounded $W^{1,2}$ norm but for which, however, every sequence of Coulomb moving frame is not bounded in $W^{1,2}$. Nevertheless we still believe that the following holds true:

Conjecture For every $k \leq m$, for every $n \in \mathbb{N}$ for every N^k, k-dimensional closed submanifold of $\mathbb{R}^m$, and for every $C > 0$ there exists $\delta(C, n, N^k) > 0$ such that if u is a $W^{1,2}$ weakly harmonic map from $B^n_2(0)$ into N^k satisfying

$$\int_{B^n_2(0)} |\nabla u|^2 \leq C, \tag{I.30}$$

then

$$\int_{B_1^n(0)} |\nabla^2 u| \leq \delta(C, n, N^k). \tag{I.31}$$

Such an estimate is known when C is small enough. The existence of such an estimate for arbitrary C would, in particular, permit to extend the quantization result of [LiR] to general targets.

Finally, in general dimension, the following conservation laws generalizing (I.28) should play an important role in the theory of weakly harmonic maps:

Theorem I.8 *Let u be a $W^{1,2}$ weakly harmonic map from B^n into N^k, a closed oriented C^2 k-dimensional submanifold of $\mathbb{R}^m$. Let $(e_1, \dots, e_k)$ be a Coulomb moving frame associated to u (the map $x \to (e_1, \dots, e_k)$ is in $W^{1,2}$, for almost every x, $(e_1(x), \dots, e_k(x))$ is an orthonormal basis of $T_{u(x)}N^k$ and $\forall i, j \in \{1, \dots, k\}$ $d^*(e_i, de_j) = 0$.) Denote $\Omega = (\Omega_i^j) \in L^2(B^n, so(k) \otimes \wedge^1\mathbb{R}^n)$ the connection given by*

$$\Omega_i^j := (e_j, de_i).$$

Let $\Phi \in L^4(B^n, M_k(\mathbb{R})) \cap W^{1,2}$ and $\Psi \in L^4(B^n, M_k(\mathbb{R}) \otimes \wedge^2\mathbb{R}^n) \cap W^{1,2}$ solving the linear equation

$$d_\Omega \Phi + d_\Omega^* \Psi = 0, \tag{I.32}$$

where $(d_\Omega \Phi)_i^j := d\Phi_i^j + \Phi_i^k \wedge \Omega_k^j$ and d_Ω^* is the adjoint of d_Ω given by $(d_\Omega^* \Psi)_i^j := d^*\Psi_i^j + *(*\Psi_i^k \wedge \Omega_k^j)$. Then the following conservation laws are satisfied

$$d\left(*\Phi(du, e) + (-1)^{n-1}(*\Psi) \wedge (du, e)\right) = 0, \tag{I.33}$$

where (du, e) is the element in $L^2(B^n, \mathbb{R}^k \otimes \wedge^1\mathbb{R}^n)$ given by $\{(du, e_j)\}_{j=1,\dots,k}$ and $\Phi(du, e)$ and $*\Psi \wedge (du, e)$ denote respectively the elements in $L^{\frac{4}{3}}(B^n, \mathbb{R}^k \otimes \wedge^1\mathbb{R}^n)$ and in $L^{\frac{4}{3}}(B^n, \mathbb{R}^k \otimes \wedge^{n-1}\mathbb{R}^n)$ given by $\Phi_i^j(du, e_j)$ and $*\Psi_i^j \wedge (du, e_j)$.

Observe that (I.33) generalizes (I.28) to general dimension by taking

$$\Phi = \cosh a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Psi = \sinh a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dx \wedge dy. \tag{I.34}$$

Existence of Coulomb moving frames is discussed in [Hel] and is proved under the assumption that N^k is sufficiently regular and modulo some isometric embeddings in a submanifold diffeomorphic to a torus. Again here, under the assumption that $\|\Omega\|_{M_2^1}$ is below some positive constant depending only on n and k, the elliptic linear system (I.32) becomes critical and the existence of Φ and Ψ solving (I.32) should be looked for in the space M^2_2 which embeds in L^4 in every dimension.

The paper is organised as follows: in Sect. 2 we prove Theorem I.1 to Theorem I.6. In Sect. 3 we prove Theorem I.5. In Sect. 4 we prove Theorem I.7 and Theorem I.8. In the appendix we recall Wente's result and establish several lemmas used in Sects. 2 and 3.

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II Proof of Theorems I.1–I.6

II.1 Proof of Theorem I.4. Let $\varepsilon(m) > 0$ given by Lemma A.3. Let $\Omega \in L^2(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ satisfying

$$\int_{D^2} |\Omega|^2 \leq \varepsilon(m). \tag{II.1}$$

Let then $P \in W^{1,2}(D^2, SO(m))$ and $\xi \in W^{1,2}(D^2, so(m))$ given by Lemma A.3 satisfying

$$\nabla^{\perp}\xi = P^{-1}\nabla P + P^{-1}\Omega P \quad \text{in } D^2, \tag{II.2}$$

with $\xi = 0$ on ∂D^2 and such that

$$\|\xi\|_{W^{1,2}} + \|P\|_{W^{1,2}} + \|P^{-1}\|_{W^{1,2}} \leq C(m)\ \|\Omega\|_{L^2}. \tag{II.3}$$

We look for A and B solving (I.18) and introducing $\tilde{A} := AP$, it means that we are looking for $\tilde{A}$ and B solving

$$\nabla\tilde{A} - \nabla^{\perp}BP = \tilde{A}\nabla^{\perp}\xi. \tag{II.4}$$

First we aim to solve the following system ($\tilde{A}$ will be chosen to be $\hat{A} + id_m$ later)

$$\begin{cases} \Delta\hat{A} = \nabla\hat{A}\cdot\nabla^{\perp}\xi + \nabla^{\perp}B\cdot\nabla P & \text{in } D^2, \\ \Delta B = \nabla^{\perp}\hat{A}\cdot\nabla P^{-1} - div(\hat{A}\nabla\xi P^{-1}) - div(\nabla\xi P^{-1}) & \text{in } D^2, \\ \dfrac{\partial\hat{A}}{\partial\nu} = 0 \quad \text{and} \quad B = 0 & \text{on } \partial D^2, \\ \text{and } \displaystyle\int_{D^2}\hat{A} = 0 \end{cases} \tag{II.5}$$

for $\hat{A} \in M_m(\mathbb{R})$ and $B \in M_m(\mathbb{R})$. Observing that the right-hand-side of the first equation of (II.5) is made of jacobians: $(\nabla\hat{A}\cdot\nabla^{\perp}\xi)^j_i = \partial_y\hat{A}^k_i\,\partial_x\xi^j_k - \partial_x\hat{A}^k_i\,\partial_y\xi^j_k$ and $-(\nabla^{\perp}B\cdot\nabla P)^j_i = \partial_x B^k_i\,\partial_y P^j_k - \partial_y B^k_i\,\partial_x P^j_k$, using Lemma A.1, standard elliptic estimates and the fact that $P \in SO(m)$, we have the a-priori estimates

$$\|\hat{A}\|_{W^{1,2}} + \|\hat{A}\|_{L^\infty} \leq C\ \|\xi\|_{W^{1,2}}\ \|\hat{A}\|_{W^{1,2}} + C\ \|P\|_{W^{1,2}}\ \|B\|_{W^{1,2}}, \tag{II.6}$$

$$\|B\|_{W^{1,2}} \leq C\ \|P^{-1}\|_{W^{1,2}}\ \|\hat{A}\|_{W^{1,2}} + C\ \|\xi\|_{W^{1,2}}\ \|\hat{A}\|_{L^\infty} + C\|\xi\|_{W^{1,2}}. \tag{II.7}$$

Thus for $\|\Omega\|_{L^2}$ small enough, using a standard fixed point argument, we obtain the existence of $\hat{A}$ and B satisfying (II.5) and

$$\|\hat{A}\|_{W^{1,2}} + \|\hat{A}\|_{L^\infty} + \|B\|_{W^{1,2}} \leq C\ \|\Omega\|_{L^2}. \tag{II.8}$$

(Observe that, using the result of [CLMS], we even have $\|\hat{A}\|_{W^{2,1}} \leq C\ \|\Omega\|_{L^2}$). Let now $\tilde{A} := \hat{A} + id_m$. From the first equation of (II.5) we obtain the existence of C in $W^{1,2}$ satisfying

$$\nabla\tilde{A} - \tilde{A}\,\nabla^\perp\xi - \nabla^\perp BP = \nabla^\perp C. \tag{II.9}$$

Moreover, up to addition of a constant, C satisfies

$$\begin{cases} div(\nabla C P^{-1}) = 0 & \text{in } D^2, \\ C = 0 & \text{on } \partial D^2. \end{cases} \tag{II.10}$$

Using now Lemma A.2, we obtain that C is identically zero and $A := \tilde{A}P^{-1}$ and B satisfy (I.21), (I.22) and (I.23). Theorem I.4 is then proved.

II.2 Proof of Theorem I.3. Theorem I.3 follows from a direct computation.

II.3 Proof of Theorem I.1. Since the desired result is local (continuity of u), we can always assume that $\int_{D^2} |\Omega|^2 \leq \varepsilon(m)$ where $\varepsilon(m)$ is given by Theorem I.4. Moreover, let A and B given by Theorem I.4. From Theorem I.4 they solve the following system

$$\begin{cases} div(A\,\nabla u) = -\nabla B \cdot \nabla^\perp u, \\ curl(A\,\nabla u) = \nabla^\perp A \cdot \nabla u. \end{cases} \tag{II.11}$$

Using standard elliptic estimates, we get the existence of E and D in $W^{1,2}(D^2)$ such that

$$A\,\nabla u = \nabla^\perp E + \nabla D. \tag{II.12}$$

Moreover, using the jacobian structure of the right-hand-sides of the equations in (II.11), the results in [CLMS] imply that E and D are in $W^{2,1}$ on the disk of half radius $D^2_{1/2}$. Therefore $\nabla u = A^{-1}\nabla^\perp E + A^{-1}\nabla D$ is in $W^{1,1}$ on this disk. Using the embedding of $W^{2,1}$ into C^0 in 2 dimension we get the desired result and Theorem I.1 is proved.

II.4 Proof of Theorem I.2. Let N^k be a k-dimensional submanifold of $\mathbb{R}^m$. Let π_N be the orthogonal projection on N^k defined in a small tubular neighborhood of N^k. Let ω be a 2-form on N^k and let $\tilde{\omega}$ be the pull back of ω by π_N in this small tubular neighborhood: $\tilde{\omega} := \pi_N^*\omega$. Following [Hel] Chap. 4, critical points of (I.13) in $W^{1,2}(D^2, N^k)$ satisfy the following Euler–Lagrange equation

$$\Delta u^i + A^i(u)_{j,l}\nabla u^j \cdot \nabla u^l + \lambda(u)^i_{j,l}\partial_x u^j \partial_y u^l = 0, \tag{II.13}$$

where $\lambda(u)^i_{j,l} := d\tilde{\omega}_u(\varepsilon_i, \varepsilon_j, \varepsilon_l)$ where $(\varepsilon_l)_{l=1,\dots,m}$ is the canonical basis of $\mathbb{R}^m$. Thus, in particular, $\lambda(u)^i_{j,l} = -\lambda(u)^j_{i,l}$. Since $(A^j_{i,l})_{j=1,\dots,m} = A(\varepsilon_i, \varepsilon_l)$ is perpendicular to $T_u N^k$ for every i and l, we have that

$$\forall i, l \in \{1, \dots, m\} \qquad \sum_j A^j_{i,l}\nabla u^j = 0. \tag{II.14}$$

Thus finally (II.13) becomes

$$\begin{aligned} \Delta u^i &+ \left[A^i(u)_{j,l} - A^j(u)_{i,l}\right]\nabla u^l \cdot \nabla u^j \\ &+ \frac{1}{4}\left[\lambda(u)^i_{j,l} - \lambda(u)^j_{i,l}\right]\nabla^\perp u^l \cdot \nabla u^j = 0. \end{aligned} \tag{II.15}$$

Introducing $\Omega := (\Omega^j_i)_{i,j}$ where

$$\Omega^i_j := \left[A^i(u)_{j,l} - A^j(u)_{i,l}\right]\nabla u^l + \frac{1}{4}\left[\lambda(u)^i_{j,l} - \lambda(u)^j_{i,l}\right]\nabla^\perp u^l, \tag{II.16}$$

we have succeeded in writing $W^{1,2}$ critical points of lagrangian of the form (I.13) as solutions to PDE of the form (I.7) for some $\Omega \in L^2(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$. Theorem I.2 is then proved.

III Conservation laws and passage to the limit in PDEs

The goal of this section is to prove Theorem I.5.

Let $\Omega_n \in L^2(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ converging weakly to some Ω and f_n and u_n respectively converging to zero in $H^{-1}(D^2, \mathbb{R}^m)$ and bounded in $W^{1,2}(D^2, \mathbb{R}^m)$. We can always assume that u_n converges weakly to some u in $W^{1,2}(D^2, \mathbb{R}^m)$. Let $\lambda < 1$ and let $\varepsilon(m)$ given by Theorem I.4. To every x in $B^2_\lambda(0)$ we assign $r_{x,n} \le 1 - |x|$ such that $\int_{B_{r_{x,n}}(x)} |\Omega_n|^2 = \varepsilon(m)$ or $r_{x,n} = 1 - |x|$ in case $\int_{B_{1-|x|}(x)} |\Omega_n|^2 < \varepsilon(m)$. $\{B_{r_x}(x)\}$ for every x in $B^2_\lambda(0)$ realizes of course a covering of $B^2_\lambda(0)$. We extract a Vitali covering from it which ensures that every point in $B^2_\lambda(0)$ is covered by a number of balls bounded by a universal number. Since $\int_{D^2} |\Omega_n|^2$ is uniformly bounded, the number of balls in each such a Vitali covering for each n is also uniformly bounded and, modulo extraction of a subsequence, we can assume that it is fixed and equal to N independent of n. Let $\{B_{r_{i,n}}(x_{i,n})\}_{i=1,\dots,N}$ be this covering. Modulo extraction of a subsequence we can always assume that each sequence $x_{i,n}$ converges in $\overline{B}^2_\lambda(0)$ to a limit x_i and that each sequence $r_{i,n}$ converges to a non negative number r_i (which could be zero of course). We claim that equation (I.7) is satisfied on each $B_{r_i}(x_i)$. Let $A_{i,n}$ and $B_{i,n}$ given by Theorem I.4 in $B_{r_{i,n}}(x_{i,n})$ for Ω_n. We then have

$$div\Big(A_{i,n}\nabla u_n + B_{i,n}\nabla^\perp u_n\Big) = -A_{i,n} f_n \quad \text{ in } B_{r_{i,n}}(x_{i,n}), \tag{III.1}$$

where $A_{i,n}$ and $B_{i,n}$ satisfy

$$\nabla A_{i,n} - A_{i,n}\Omega_{i,n} = \nabla^\perp B_{i,n}. \tag{III.2}$$

We can extract a subsequence such that each of the couples $(A_{i,n}, B_{i,n})$ weakly converge in $W^{1,2}$ to some limit (A_i, B_i) in every $B_{r_i}(x_i)$. Because of the weak convergences in $W^{1,2}$ we have strong convergences in L^2 and then we have that

$$A_{i,n}\nabla u_n + B_{i,n}\nabla^\perp u_n \longrightarrow A_i\nabla u + B_i\nabla^\perp u \quad \text{ in } \mathcal{D}', \tag{III.3}$$

$$\nabla A_{i,n} - A_{i,n}\Omega_{i,n} - \nabla^\perp B_{i,n} \longrightarrow \nabla A_i - A_i\Omega - \nabla^\perp B_i \quad \text{ in } \mathcal{D}', \tag{III.4}$$

and

$$-A_{i,n}\ f_n \to 0 \quad \text{ in } \mathcal{D}'. \tag{III.5}$$

Combining then (III.1), ... ,(III.5) we obtain that

$$div\Big(A_i\nabla u + B_i\nabla^\perp u\Big) = 0 \quad \text{ in } B_{r_i}(x_i), \tag{III.6}$$

and that

$$\nabla A_i - A_i\Omega = \nabla^\perp B_i \quad \text{ in } B_{r_i}(x_i). \tag{III.7}$$

Combining (III.6) and (III.7) we then have that

$$A_i\,[\Delta u + \Omega\cdot\nabla u] = 0 \quad \text{ in } B_{r_i}(x_i). \tag{III.8}$$

From (I.21), because of the pointwise convergence of $A_{i,n}$, we get the invertibility of A_i and (III.8) implies that

$$\Delta u + \Omega\cdot\nabla u = 0 \quad \text{ in } B_{r_i}(x_i). \tag{III.9}$$

It is clear that every point in $B^2_\lambda(0)$ is in the closure of the union of the $B_{r_i}(x_i)$. Let x be a point which is none of the $B_{r_i}(x_i)$. It seats then on the circle, boundary of one of the $B_{r_i}(x_i)$. For convexity reason, it has to seat at the boundary of at least 2 different circles. 2 different circles can intersect at only finitely many points (0,1 or 2 points), since there are finitely many circles, only finitely many points in $B^2_\lambda(0)$ can be outside the union of the $B_{r_i}(x_i)$. Thus the distribution $\Delta u + \Omega\cdot\nabla u$ is supported at at mostly finitely many points. Since $\Delta u + \Omega\cdot\nabla u \in H^{-1} + L^1$ it is identically zero on $B^2_\lambda(0)$. Since this holds for every $\lambda < 1$ we have proved Theorem I.5.

IV Conservation laws and moving frames

IV.1 Proof of Theorem I.7. First (I.28) is the result of a direct computation, granting the fact that $(\Delta u, e) = 0$. It remains then to prove (I.29). We rewrite (I.28) in the form (using $\mathbb{Z}_2$ indexation)

$$\begin{cases} div(\cosh a(\nabla u, e_i)) = (-1)^i(\nabla(\sinh a\, e_{i+1}), \nabla^\perp u), \\ curl(\cosh a(\nabla u, e_i)) = (\nabla u, \nabla^\perp(\cosh a\, e_i)). \end{cases} \tag{IV.1}$$

Using then Lemma A.1, and standard elliptic estimate, there exist $E \in W^{2,1}(D^2_{1/2}, \mathbb{R}^2)$ and $D \in W^{2,1}(D^2_{1/2}, \mathbb{R}^2)$ such that

$$\cosh a(\nabla u, e_i) = \nabla E_i + \nabla^\perp D_i, \tag{IV.2}$$

moreover

$$\begin{aligned} \|E\|_{W^{2,1}(D^2_{1/2})} + \|D\|_{W^{2,1}(D^2_{1/2})} \leq C &\|\nabla(\sinh ae)\|_{L^2}\, \|\nabla u\|_{L^2} \\ &+ \|\nabla(\cosh ae)\|_{L^2}\, \|\nabla u\|_{L^2} \\ &+ \|\cosh a(\nabla u, e)\|_{L^2}. \end{aligned} \tag{IV.3}$$

Thus we have

$$\begin{aligned} \|E\|_{W^{2,1}(D^2_{1/2})} + \|D\|_{W^{2,1}(D^2_{1/2})} \leq C\, &e^{\|a\|_\infty}\, \|\nabla e\|_{L^2}\, \|\nabla u\|_{L^2} \\ &+ e^{\|a\|_\infty}\, \|\nabla a\|_{L^2}\, \|\nabla u\|_{L^2} + e^{\|a\|_\infty}\, \|\nabla u\|_{L^2}. \end{aligned} \tag{IV.4}$$

Applying Lemma A.1 (with the optimal constants given in [Hel]) to (I.27) we have

$$\|a\|_{L^\infty(D^2)} \leq \frac{1}{2\pi}\|\nabla e_1\|_{L^2}\, \|\nabla e_2\|_{L^2}, \tag{IV.5}$$

and

$$\|\nabla a\|_{L^2(D^2)} \leq \frac{1}{\sqrt{2\pi}}\|\nabla e_1\|_{L^2}\, \|\nabla e_2\|_{L^2}. \tag{IV.6}$$

We rewrite (IV.2) in the form

$$\nabla u = (\cosh a)^{-1}\left(\nabla E_j e_j + \nabla^\perp D_j e_j\right). \tag{IV.7}$$

Combining then (IV.4), (IV.5) and (IV.7) we get (I.29) and Theorem I.7 is proved.

IV.2 Proof of Theorem I.8. Theorem I.8 follows from a direct computation.

A Appendix

Lemma A.1 [We,BrC,Ta1,CLMS] *Let a and b in $L^1(D^2, \mathbb{R})$ such that ∇a and ∇b are in $L^2(D^2)$. Let φ be the solution of*

$$\begin{cases} \Delta\varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} & \text{in } D^2, \\ \varphi = 0 \quad \text{or} \quad \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial D^2. \end{cases} \tag{A.1}$$

Then the following estimates hold

$$\|\varphi\|_{L^\infty(D^2)} + \|\nabla\varphi\|_{L^2(D^2)} + \|\nabla^2\varphi\|_{L^1(D^2)} \le \|\nabla a\|_{L^2(D^2)}\,\|\nabla b\|_{L^2(D^2)}, \tag{A.2}$$

where we choose $\int_{D^2}\varphi = 0$ for the Neumann boundary data.

Lemma A.2 *There exists $\varepsilon > 0$ such that for every $P \in W^{1,2}(D^2, Gl(m))$ satisfying*

$$\int_{D^2} |\nabla P|^2 + |\nabla P^{-1}|^2 \le \varepsilon, \tag{A.3}$$

then, $C \equiv 0$ is the unique solution in $W^{1,2}(D^2, M_m(\mathbb{R}))$ of the following problem

$$\begin{cases} div(\nabla C\, P^{-1}) = 0 & \text{in } D^2, \\ C = 0 & \text{on } \partial D^2. \end{cases} \tag{A.4}$$

Proof of Lemma A.2 From (A.4) there exists $D \in W^{1,2}(D^2, M_m(\mathbb{R}))$ such that $\nabla^\perp D = \nabla C\, P^{-1}$ and we can choose D such that $\int_{D^2} D = 0$. Thus C and D satisfy respectively

$$\begin{cases} \Delta C = \nabla^\perp D \cdot \nabla P & \text{in } D^2, \\ C = 0 & \text{on } \partial D^2, \end{cases} \tag{A.5}$$

and

$$\begin{cases} \Delta D = -\nabla^\perp C \cdot \nabla P^{-1} & \text{in } D^2, \\ \dfrac{\partial D}{\partial \nu} = 0 & \text{on } \partial D^2. \end{cases} \tag{A.6}$$

Thus, using Lemma A.1 and (A.3), for ε small enough, we have

$$\begin{aligned} &\|\nabla C\|_{L^2(D^2)} \le \frac{1}{2}\|\nabla D\|_{L^2(D^2)} \quad \text{and} \\ &\|\nabla D\|_{L^2(D^2)} \le \frac{1}{2}\|\nabla C\|_{L^2(D^2)}, \end{aligned} \tag{A.7}$$

which implies that $C \equiv 0$ and $D \equiv 0$ and Lemma A.2 is proved.

Lemma A.3 *There exist* $\varepsilon(m) > 0$ *and* $C(m) > 0$ *such that for every* Ω *in* $L^2(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ *satisfying*

$$\int_{D^2} |\Omega|^2 \leq \varepsilon(m), \tag{A.8}$$

then there exist $\xi \in W^{1,2}(D^2, so(m))$ *and* $P \in W^{1,2}(D^2, SO(m))$ *such that*

i)

$$\nabla^\perp \xi = P^{-1}\nabla P + P^{-1}\Omega P \quad \textit{in } D^2, \tag{A.9}$$

ii)

$$\xi = 0 \quad \textit{on } \partial D^2, \tag{A.10}$$

iii)

$$\|\xi\|_{W^{1,2}(D^2)} + \|P\|_{W^{1,2}(D^2)} \leq C(m)\ \|\Omega\|_{L^2(D^2)}. \tag{A.11}$$

In order to prove Lemma A.3 we follow the strategy of [Uhl] and we first prove the following result.

Lemma A.4 *There exist* $\varepsilon(m) > 0$ *and* $C(m) > 0$ *such that for every* Ω *in* $W^{1,2}(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ *satisfying*

$$\int_{D^2} |\Omega|^2 \leq \varepsilon(m), \tag{A.12}$$

then there exist $\xi \in W^{2,2}(D^2, so(m))$ *and* $P \in W^{2,2}(D^2, SO(m))$ *such that*

i)

$$\nabla^\perp \xi = P^{-1}\nabla P + P^{-1}\Omega P \quad \textit{in } D^2, \tag{A.13}$$

ii)

$$\xi = 0 \quad \textit{on } \partial D^2, \tag{A.14}$$

iii)

$$\|\xi\|_{W^{1,2}(D^2)} + \|P\|_{W^{1,2}(D^2)} \leq C(m)\ \|\Omega\|_{L^2(D^2)}, \tag{A.15}$$

iv)

$$\|\xi\|_{W^{2,2}(D^2)} + \|P\|_{W^{2,2}(D^2)} \leq C(m)\ \|\Omega\|_{W^{1,2}(D^2)}. \tag{A.16}$$

Proof of Lemma A.4 $\Longrightarrow$ *Lemma A.3* Let Ω in $L^2(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ satisfying (A.8). Introduce Ω_k in $W^{1,2}(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ converging strongly in L^2 to Ω. Let ξ_k and P_k satisfying (A.13), ... , (A.16) for Ω_k. Because of (A.15) there exists a subsequence $\xi_{k'}$ and $P_{k'}$ converging weakly in $W^{1,2}$ to ξ and P. Weak convergence in $W^{1,2}$ implies almost everywhere convergence of $P_{k'}$ to P. Since $P_{k'}^t P_{k'} = id_m$, this equation passes to the limit and we have that $P \in W^{1,2}(D^2, SO(m))$. Moreover, from Rellich compact embedding, $P_{k'}$ converges strongly in every L^q ($q < +\infty$) and $P^t_{k'}\Omega_{k'}P_{k'}$ respectively $P^t_{k'}\nabla P_{k'}$ converge in distribution sense to $P^t\Omega P$ and respectively $P^t\nabla P$. Therefore (A.9) is satisfied at the limit. By continuity of the trace (A.10) is also satisfied. Finally, by lower-semicontinuity of the $W^{1,2}$ norm with respect to the weak $W^{1,2}$ convergence, we also obtain (A.11) and Lemma A.3 is proved.

Proof of Lemma A.4 We follow the strategy in [Uhl]. We introduce the set

$$\mathcal{U}_{\varepsilon,C} = \left\{ \begin{array}{l} \Omega \in W^{1,2}(D^2, so(m) \otimes \wedge^1\mathbb{R}^2) \text{ satisfying } \displaystyle\int_{D^2} |\Omega|^2 \leq \varepsilon, \\ \text{and for which there exists } \xi \in W^{2,2}(D^2, so(m)), \\ \text{and } P \in W^{2,2}(D^2, SO(m)) \text{ solving } (A.13), \ldots, (A.16) \end{array} \right\}. \tag{A.17}$$

The previous argument can be adapted to show that $\mathcal{U}_{\varepsilon,C}$ is closed. We now establish the following assertion:

Claim For any fixed C there exists ε small enough, such that, for any Ω in $\mathcal{U}_{\varepsilon,C}$ there exists a neighborhood of Ω in $W^{1,2}$ included in $\mathcal{U}_{\varepsilon,C}$.

Proof of the claim Let $\Omega \in \mathcal{U}_{\varepsilon,C}$ satisfying $\int_{D^2} |\Omega|^2 < \varepsilon$. Let ξ and P satisfying (A.13), ... , (A.16) for Ω. Following the arguments in [Uhl, Lemmas 2.7 and 2.8], for every $\alpha > 0$ we can find $\delta > 0$ such that, for every $\lambda \in W^{1,2}(D^2, so(m) \otimes \wedge^1\mathbb{R})$ satisfying $\|\lambda\|_{W^{1,2}} \leq \delta$, there exists $\xi_\lambda \in W^{2,2}(D^2, so(m))$ and $Q_\lambda \in W^{2,2}(D^2, SO(m))$ satisfying

$$\begin{cases} \nabla^\perp\xi_\lambda = Q_\lambda^{-1}\nabla Q_\lambda + Q_\lambda^{-1}(\nabla^\perp\xi + \lambda)Q_\lambda & \text{in } D^2, \\ \xi_\lambda = 0 & \text{on } \partial D^2, \end{cases} \tag{A.18}$$

and

$$\|Q_\lambda - Id_m\|_{W^{2,2}} + \|\xi_\lambda - \xi\|_{W^{2,2}} \leq \alpha. \tag{A.19}$$

From (A.13) and (A.18) we then have

$$\nabla^\perp\xi_\lambda = (PQ_\lambda)^{-1}\nabla(PQ_\lambda) + (PQ_\lambda)^{-1}(\Omega + P\lambda P^{-1})PQ_\lambda. \tag{A.20}$$

Since $P \in W^{2,2}$ the map $\lambda \to P\lambda P^{-1}$ and its inverse $\zeta \to P^{-1}\zeta P$ are continuous from $W^{1,2}$ into $W^{1,2}$ (using the fact that $W^{2,2}$ embeds in $W^{1,4}$ in dimension 2 and that $P \in SO(m)$). Therefore, for every $\beta > 0$, there exists $\eta > 0$ such that for every $\zeta \in W^{1,2}(D^2, so(m) \otimes \wedge^1\mathbb{R})$ satisfying $\|\zeta\|_{W^{1,2}} \leq \eta$, there exists $R \in W^{2,2}(D^2, SO(m))$ and $\nu \in W^{2,2}(D^2, M_m(\mathbb{R}))$ such that

$$\begin{cases} \nabla^\perp\nu = R^{-1}\nabla R + R^{-1}(\Omega + \zeta)R & \text{in } D^2, \\ \nu = 0 & \text{on } \partial D^2. \end{cases} \tag{A.21}$$

Moreover we have

$$\|R - P\|_{W^{2,2}} + \|\nu - \xi\|_{W^{2,2}} < \beta. \tag{A.22}$$

Considering now $\beta < \varepsilon^{\frac{1}{2}}$, since $\|P\|_{W^{1,2}} + \|\xi\|_{W^{1,2}} \leq C\varepsilon^{\frac{1}{2}}$ ((A.15) is satisfied for $\Omega \in \mathcal{U}_{\varepsilon,C}$), we have

$$\|R\|_{W^{1,2}} + \|\nu\|_{W^{1,2}} \leq (C+1)\varepsilon^{\frac{1}{2}}. \tag{A.23}$$

In order to complete the proof of claim it remains to establish (A.15) and (A.16), providing that ε has been chosen small enough. This will be a consequence of the following lemma.

Lemma A.5 *There exist $C(m) > 0$ ε and $\delta > 0$ such that for every $P \in W^{2,2}(D^2, SO(m))$ and $\xi \in W^{2,2}(D^2, so(m))$ satisfying* (A.13) *and* (A.14) *for some $\Omega \in W^{1,2}(D^2, so(m))$ satisfying $\int_{D^2} |\Omega|^2 \leq \varepsilon$, if*

$$\|P\|_{W^{1,2}} + \|\xi\|_{W^{1,2}} \leq \delta, \tag{A.24}$$

then (A.15) *and* (A.16) *are satisfied.*

Proof of Lemma A.5 We first establish the critical estimate (A.15).

(A.13) and (A.14) imply that ξ solves the following elliptic PDE

$$\begin{cases} -\Delta\xi = \nabla P^t \cdot \nabla^\perp P + div(P^t \Omega P) & \text{in } D^2, \\ \xi = 0 & \text{on } \partial D^2. \end{cases} \tag{A.25}$$

Using Lemma A.1 and standard elliptic PDE we have

$$\|\nabla\xi\|_{L^2} \leq C\ \|\nabla P^t\|_{L^2}\|\nabla P\|_{L^2} + C\|\Omega\|_{L^2}. \tag{A.26}$$

Using the hypothesis that $\|\nabla P\|_{L^2} \leq \delta$ we have that

$$\|\nabla\xi\|_{L^2} \leq C\ \delta\ \|\nabla P\|_{L^2} + C\|\Omega\|_{L^2}. \tag{A.27}$$

From (A.13) we have that

$$\|\nabla P\|_{L^2} \leq 2\|\nabla\xi\|_{L^2} + 2\|\Omega\|_{L^2}. \tag{A.28}$$

Combining (A.27) and (A.28) we have then

$$\|\nabla\xi\|_{W^{1,2}} \leq 2C\ \delta\ \|\nabla\xi\|_{L^2} + (C + 2C\ \delta)\|\Omega\|_{L^2}. \tag{A.29}$$

Choosing then $2C\ \delta < 1/2$ we obtain inequality (A.15).

It remains to establish (A.16). From (A.25) again, using standard elliptic estimates and the embedding of $W^{1,1}$ into L^2 in 2 dimensions, we have

$$\begin{aligned} \|\xi\|_{W^{2,2}} \leq\ & C\|\nabla P^t \cdot \nabla^\perp P\|_{W^{1,1}} + C\|\Omega\|_{W^{1,2}} \\ & + C\|\nabla P^t \Omega\|_{L^2} + C\|\Omega\nabla P\|_{L^2}. \end{aligned} \tag{A.30}$$

Using Cauchy-Schwarz we have first

$$\|\nabla P^t \cdot \nabla^\perp P\|_{W^{1,1}} \leq C\|P\|_{W^{2,2}}\|P\|_{W^{1,2}}. \tag{A.31}$$

Using the embedding of $W^{1,1}$ in L^2 and Cauchy-Schwarz we have

$$\begin{aligned}\|\nabla P^t\Omega\|_{L^2} + \|\Omega\nabla P\|_{L^2} &\leq C\|\nabla P^t\Omega\|_{W^{1,1}} + \|\Omega\nabla P\|_{W^{1,1}}\\ &\leq C\|P\|_{W^{2,2}}\|\Omega\|_{L^2} + C\|\Omega\|_{W^{1,2}}\|P\|_{W^{1,2}}.\end{aligned} \tag{A.32}$$

Combining (A.30), (A.31) and (A.32) we have then that

$$\|\xi\|_{W^{2,2}} \leq C(\delta + \varepsilon^{\frac{1}{2}})\ \|P\|_{W^{2,2}} + C\|\Omega\|_{W^{1,2}}. \tag{A.33}$$

Using now (A.13), we have

$$\begin{aligned}\|\nabla P\|_{W^{1,2}} &\leq C\|\xi\|_{W^{2,2}} + C\||\nabla^\perp\xi||\nabla^2 P|\|_{L^1}\\ &\quad + C\||\nabla^2\xi||\nabla P|\|_{L^1} + C\|\Omega\|_{W^{1,2}} + C\|\nabla P\Omega\|_{W^{1,1}}\\ &\leq C(1+\delta)\|\xi\|_{W^{2,2}} + C(1+\delta)\|\Omega\|_{W^{1,2}} + C(\delta + \varepsilon^{\frac{1}{2}})\ \|P\|_{W^{2,2}}.\end{aligned} \tag{A.34}$$

Combining (A.33) and (A.34), for $C(\delta + \varepsilon^{\frac{1}{2}}) < 1/2$ we obtain estimate (A.16) and Lemma A.5 is proved.

End of the proof of Lemma A.4 Let Ω in $W^{1,2}(D^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ satisfying $\int_{D^2}|\Omega|^2 < \varepsilon$ for ε for which claim holds. We consider the path $\Omega_t = \phi_t^*\Omega$ where $\phi_t(x) = tx$ and $t \in [0,1]$. Since $\int_{D^2}|\Omega_t|^2 = \int_{D_t^2}|\Omega|^2$ is an increasing function of t we have then a path among the elements in $W^{1,2}(D^2, so(m) \otimes \mathbb{R}^2)$ satisfying $\int_{D^2}|\Omega_t|^2 \leq \varepsilon$ connecting 0 and Ω. Using the closeness of $\mathcal{U}_{\varepsilon,C}$, the openness property given by claim and the fact that $0 \in \mathcal{U}_{\varepsilon,C}$, by the mean of a standard continuity argument we obtain that Ω is in $\mathcal{U}_{\varepsilon,C}$ and Lemma A.4 is proved.

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